TECHNIQUES IN NON-ABELIAN ADDITIVE COMBINATORICS, EXAMPLES SHEET 1

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General remark: I have not redone all these questions, and therefore some of the bounds I state may be a little approximate. If you get different, but similar bounds, then don't worry!

1. The subset $\{0,1\}^n$ of $\mathbb{F}_3^n$ has density $(2/3)^n$ and contains no affine line. Can you improve on this bound?

2. (i) For each triple of positive integers m, r and d, let $S_{m,r,d}$ be the set of all points $x=(x_1,\dots,x_d)\in\{0,1,\dots,m-1\}^d$ such that $\sum_i x_i^2=r$. For each d and m prove that there exists r such that $|S_{m,r,d}|\geq m^{d-2}/d$.

(ii) Prove that no $S_{m,r,d}$ contains a non-trivial triple (x,y,z) with $x+z=2y$.

(iii) By considering a suitable projection of a suitable set $S_{m,r,d}$, deduce that for each n there exists a set $A\subset\{1,\dots,n\}$ of density at least $\exp(-c\sqrt{\log n})$ (where c is some absolute constant) that contains no arithmetic progression of length 3.

[The above example was discovered by Behrend in 1947. The bound has stayed almost still since then.]

3. Let A and B be subsets of Abelian groups. We say that they are *Freiman isomorphic of order k* if there is a bijection $\phi:A\to B$ such that the equation

$$\phi(a_1)+\dots+\phi(a_k)=\phi(a_1')+\dots+\phi(a_k')$$

holds for elements of A if and only if the equation

$$a_1+\dots+a_k=a_1'+\dots+a_k'$$

holds. Prove that if A and B are subsets of $\mathbb{F}_p^n$ that are Freiman isomorphic of order 8, then $2A-2A$ contains a subspace of dimension d if and only if $2B-2B$ contains a subspace of dimension d.

4. Let A be a subset of $\mathbb{F}_p^n$ and suppose that $|8A-8A|\leq C|A|$. Let X be a subspace of $\mathbb{F}_p^n$ of codimension d, chosen uniformly at random from all such subspaces, and let P_X

be the quotient map from $\mathbb{F}_p^n$ to $\mathbb{F}_p^n/X$. Prove that as long as $p^d > C$, there is a non-zero probability that $P_X(A)$ and A are Freiman isomorphic of order 8. Use this result to prove that $2A - 2A$ contains a large subspace (with a meaning of "large" that you should work out for yourself).

5. Use the inequality $[A_1, A_2, A_3, A_4] \leq \|A_1\|_\Box \|A_2\|_\Box \|A_3\|_\Box \|A_4\|_\Box$ to prove the inequality $|\mathbb{E}_{x,y} A(x, y)u(x)v(y)| \leq \|A\|_\Box \|u\|_2 \|v\|_2$.

6. Let $A : \mathbb{C}^n \to \mathbb{C}^m$ and let $u \in \mathbb{C}^n$ be such that $\|Au\|_2 = \|A\|_{\mathrm{op}} \|u\|_2$. Prove that if $\langle u, v\rangle = 0$, then $\langle Au, Av\rangle = 0$. Use this result to prove that A has a singular-value decomposition.

7. Use the Cauchy-Schwarz inequality several times to prove an inequality of the form

$$|\mathbb{E}_{x,y,z} f(x, y, z)a(x)b(y)c(z)| \leq \Phi(f)\|a\|_2 \|b\|_2 \|c\|_2,$$

where Φ is a functional that you should determine. (If you get the right one, it is in fact a norm, but you do not need to prove this – though it's not a bad exercise to do so.)

8. Obtain also an inequality of the form

$$|\mathbb{E}_{x,y,z} f(x, y, z)u(x, y)v(y, z)w(x, z)| \leq \Psi(f)\|u\|_\infty \|v\|_\infty \|w\|_\infty,$$

where Ψ is an obvious three-dimensional generalization of the box norm.

9. Let A be a subset of $\mathbb{F}_p^n$ of density α. What can you say about the dimension of the largest affine subspace that is contained in $A + A + A$?

10. Prove that the Bohr set $B(K, \epsilon) \subset \mathbb{Z}_N$ has density at least $(\epsilon/2)^{|K|}$. Prove also that it contains an arithmetic progression of length at least $(\epsilon/2)N^{1/|K|}$.

11. Let $A_r \subset \mathbb{F}_2^n$ be the set of all x such that $x_i = 1$ for at most r values of i. Prove that for suitably chose r (depending on n) the set A_r has density $\frac{1}{2} - o(1)$ but the set $A_r - A_r$ does not contain a subspace of bounded codimension.